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Decoherence by Lindblad motion

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Abstract

The predictive power of Lindblad equations for the dynamics of open systems is discussed. In a model with only one Lindblad operator the asymptotic state (density operator) for $t \to \infty$ determines the Lindblad operator up to an isometry. Hence the asymptotic state reached by Lindblad motion appears as an input and has to be obtained from independent physical considerations when setting up the equations of motion. To illustrate this fact we assume an asymptotic Gibbs state (and a Hamiltonian of course) and discuss the temporal behaviour of the statistical entropy. Numerical model calculations are presented which show a non-monotonic behaviour of the statistical entropy during the approach to equilibrium. As a second point in our discussion of the types of predictions derived from Lindblad equations, we show that a certain structure of the Lindblad operators leads to decoherence into superselection sectors. The latter are determined by a spectral decomposition of the Hamiltonian of the open system considered. An explicit construction of such decohering systems is given.

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1. Introduction

The derivation of equations of motion for quantum systems proceeds on different lines of argument depending on the problem considered. For the problem we are going to address—the motion of open quantum systems—it seems appropriate to work in a rather general scope of argumentation. In this sense we start by imposing general conditions to be followed by the motion of an observable B_t in the *Heisenberg picture*

$$\Theta_t : B_t|_{t=0} \longmapsto B_t|_{t>0} \qquad (1)$$

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or by the adjoint motion $\bar{\Theta}_t$ of a state ρ—the density operator—describing our quantum system in the *Schrödinger picture*[2]. As a first condition on the approach considered in this paper we postulate the Markov condition

$$\mathit{Condition}_{[1]}: \qquad \Theta_{t_1}\Theta_{t_2} = \Theta_{t_1+t_2}$$

i.e. we assume a semigroup structure for the set of motions $\{\Theta_t | t \in \mathbb{R}\}$ parametrized by the time t.

The second postulate to be imposed on quantum dynamical maps is the complete positivity of the motions Θ_t [1] which in a way expresses the notion of probability conservation in the case of open systems

$$\mathit{Condition}_{[2]}: \qquad \Theta_t \textit{ is completely positive}.$$

The Schrödinger equation (or the corresponding Liouville equation) follows from these conditions if we restrict the motions Θ_t to maps mapping pure states into pure states (for all t)[3]

$$\bar{\Theta}_t : \textit{pure state} \longmapsto \textit{pure state}. \tag{2}$$

Relaxing this restriction by allowing for pure-to-mixed state transitions

$$\bar{\Theta}_t : \textit{pure state} \longmapsto \textit{mixed state} \tag{3}$$

and assuming $\mathit{Condition}_{[1,2]}$ Lindblad [3] proved the following equations of motion,

$$\mathit{Heisenberg}: \ \partial_t B = L^B(B) \tag{4}$$

for observables and

$$\mathit{Schrödinger}: \ \partial_t \varrho = L^\varrho(\varrho) \tag{5}$$

where the generators L^B and L^ϱ are

$$L^B = \mathrm{i}[H, \cdot] + \sum_J \left(V_J^+ \cdot V_J - \frac{1}{2}\left[V_J^+ V_J, \cdot\right]_+ \right) \tag{6}$$

$$L^\varrho = -\mathrm{i}[H, \cdot] + \sum_J \left(V_J \cdot V_J^+ - \frac{1}{2}\left[V_J^+ V_J, \cdot\right]_+ \right). \tag{7}$$

We see that the very general frame set by $\mathit{Condition}_{[1,2]}$ determines the dynamics of quantum systems; as input we have in addition to the Hamiltonian H operators V_J responsible for the transition of pure states into mixed states—excluding the trivial case $V_J = 0$ or $\mathbb{I}$ all J. It is the purpose of this paper to show that a predictive scheme arises if some further specifications are introduced.

The simplest case (see [7]) arises if we restrict ourselves to only one Lindblad operator V. We furthermore assume that V^+V is invertible

$$V^+V > 0 \tag{8}$$

and commutes with the Hamiltonian

$$[H, V^+V] = 0. \tag{9}$$

[2] B and ρ are operators acting on a representation (Hilbert-) space $\mathfrak{H}$: B is bounded, $B \in \mathfrak{B}(\mathfrak{H})$, ρ is positive and trace class.

[3] The isometries resulting from the Stinespring theorem and mapping projectors $P = |\psi\rangle\langle\psi|$ (pure states) into projectors are unitaries which, written as $\exp(-\mathrm{i}Ht)$, imply the Schrödinger equation $\mathrm{i}\partial_t|\psi\rangle = H|\psi\rangle$ by $\mathit{Condition}_{[1]}$. For a detailed exposition see [3]. The set $\{\Theta_t\}$ is then the group of outer automorphisms generated by the Hamiltonian H.

These two assumptions are assumed to hold throughout the paper; in the case of more than one $V_J \neq 0, \mathbb{I}$ more complicated relations have to be formulated. The role of zero modes in V^+V will be discussed in a forthcoming paper.

We see by inspection that

$$W = (V^+V)^{-1} \tag{10}$$

is a stationary solution of the Lindblad equation (5)

$$L^{\varrho}(W) = 0. \tag{11}$$

In general we have (there are exceptional cases which are characterized by the requirement of renormalizations in subsectors and will be shown below to be the central clue for the construction of models explaining the phenomenon of decoherence for Lindblad motion)

$$\mathit{Schrödinger} : \varrho(t) \longrightarrow \varrho_{\text{stationary}} \tag{12}$$

$$\begin{aligned} &\mathit{Heisenberg} : B(t) \ \longrightarrow \ \text{trace}(B_t|_{t=0}\varrho_{\text{stationary}})\mathbb{I} \\ &t \to \infty \qquad \text{for all initial states } \varrho_t|_{t=0} \end{aligned} \tag{13}$$

where ($\mathbb{I}$ is the unit operator)

$$\varrho_{\text{stationary}} = W/\text{trace}(W). \tag{14}$$

Considering that $(W)^{-1/2}$ as given in equation (10) constitutes its absolute value we obtain the polar decomposition (see [15]) of the operator V

$$\begin{aligned} V &= U\sqrt{V^+V} \\ &= U(W)^{-1/2} \end{aligned} \tag{15}$$

where U is an isometry. We take U as unitary (i.e. we assume finite systems or additionally VV^+ invertible) and get a clear-cut physical interpretation of the equation of motion:

$\varrho_{\text{stationary}}$, taken diagonal, is a probability distribution, determined by the absolute value of the Lindblad operator V. It is approached for large times $t \to \infty$ independently of the initial state; U controls the approach to ∞.

We see in particular that the asymptotic configuration (12) plays the role of an input. It is not predicted by the Lindblad equations and has to be derived from independent physical considerations when setting up the equations of motion. The Lindblad equations predict the temporal approach to an assumed asymptotic state.

To illustrate this process we assume the system to reach thermal equilibrium and study the temporal behaviour of the statistical entropy. This study is of interest since it provides an unequivocal and, to an extent, model independent answer to the question of monotonic increase of the entropy when approaching equilibrium.

As a further demonstration of the 'predictivity' of the Lindblad scheme I shall discuss the phenomenon of decoherence in the realm of the Lindblad equation by interpreting *exceptional* cases in the asymptotics (12). We shall show that operators V_J can be constructed such that for any initial condition the motion (4) or (5) leads to the decoherence of a given quantum system into independent quantum systems. The decoherence pattern is imposed by a spectral decomposition of the Hamiltonian. This decoherence process has the character of splitting the original set of states into superselection sectors; a decomposition which although not controlled by Abelian gauge groups as in the usual definition precludes the existence of physical states as superpositions of (pure) states of separated systems. The final states of this decoherence process are the superselected projections of the final distribution (14) (properly normalized, see below). A scheme can be obtained in which superselection sectors survive decoherence as independent dynamical systems: motions in these sectors are controlled by correspondingly projected sector-Hamiltonians. We shall not pursue this construction in this paper, but indicate however that the clue here is the partial 'lindbladization' procedure described in [17].

A few remarks are in place concerning the comparision of the point of view just described to the canonical theory of the dynamics of open *systems*[4]. To recapitulate, the starting point of the canonical approach is the map $\hat{\Theta}_t$,

$$\hat{\Theta}_t : B_{t=0} \longmapsto \mathrm{tr}_{\mathrm{env}}\left(\varrho_{\mathrm{env}} \mathbb{U}_t^{+} (B_{t=0} \otimes \mathbb{I}_{\mathrm{env}}) \mathbb{U}_t\right) \tag{16}$$

where

$$\mathbb{U}_t = \exp(-\mathrm{i} H_{\mathrm{tot}} t).$$

The physical idea is the notion of an open *system* as a *system* embedded in a (larger) system—the environment—and to consider the total system as closed; (16) expresses the projection of the total dynamics on the dynamics of the open *system* (H, B are (bounded) observables of the *system* acting in $\mathfrak{H}$, ϱ_{env} is a state, $\mathbb{I}_{\mathrm{env}}$ is the unit operator on $\mathfrak{H}_{\mathrm{env}}$). The total Hamiltonian

$$H_{\mathrm{tot}} = H + H_{\mathrm{env}} + H^{\mathrm{int}}_{\mathrm{syst-env}} \tag{17}$$

describes the interaction of the *system* with its environment.

An important property of the map $\hat{\Theta}_t$ is its complete positivity; it will not however obey the Markow property $Condition_{[1]}$ in general. This in particular means that master equations derived from Zwanzig pre-master equations [4] constructed using $\hat{\theta}(t)$ and further assumptions adapted to the specific physical problems do not always respect the positivity condition. Detailed insight into this problem, in particular into its importance in actual physical applications, has been obtained in a variety of studies [2].

Implications of the structure of $\hat{\Theta}(t)$ or rather of its conjugate

$$\bar{\Theta}_t : \varrho_{\mathrm{syst}}|_{t=0} \longrightarrow \mathrm{tr}_{\mathrm{env}}(\mathbb{U}(t) \varrho_{\mathrm{syst}} \otimes \varrho_{\mathrm{env}} \mathbb{U}^{+}(t)) \tag{18}$$

for the question we are concerned with, decoherence, have been thoroughly considered. In particular, model Hamiltonians specifying these maps have been discussed in the literature: the Araki–Zurek [9, 10] Hamiltonian with a separable (factorized) *system*–environment interaction allows a proof of decoherence (for a proper definition see below) in the trace norm limit. A physically very interesting model has been discussed in [12], the model of a free particle moving on a line coupled to a massless boson field; the decisive role of the infrared divergence for decoherence (in the trace norm) has been shown. Scattering models [13] lead to interesting results in this context.

We have not yet broached the problem of a strong *system*–environment interaction when higher order perturbations in $H^{\mathrm{int}}_{\mathrm{syst-env}}$ play a role: to first order '*system*' states ϱ and '*environment*' states ϱ_{env} are well-defined, e.g. Gibbs distributions for *systems* approaching thermal equilibrium; to higher order, or even non-perturbatively, however the definition of physically well-defined environment states requires a careful (conceptual) analysis. It must be noted that the Lindblad method does not require any limitation on the 'strength' of the influence of the '*environment*' and thus on the rapidity of the approach to a final state. More precisely speaking, scaling the V_J by a factor of λ the damping term in (6) and (7) scales by a factor of λ^2, hence the (exponential) rapidity of approach scales by the same factor (as is immediately seen by the appropriate scaling of time and Hamiltonian). Two examples of physical situations where strong coupling can occur might serve as illustration.

[4] Fruitful and pleasant discussions with Joachim Kupsch on this subject are gratefully acknowledged.

- Consider a collection of atoms or molecules in an atomic or molecular matrix. Here we have periodic structures and the interaction energy is of the order of a ratio of two atomic distances: the period of the lattice and the spatial dimension of the embedded *system*, i.e. atoms or molecules. Depending on the states considered (e.g. rotational states in molecules) a high order or even non-perturbative situation can arise.
- Consider the quark–gluon plasma near the point where it separates into stable *systems*—mesons, baryons, etc. This separation point is characterized by quark–antiquark, three-quark, five-quark configurations organized in superselection sectors characterized by integer electric charges obeying the Gell-Mann–Nishijima relation. These states constitute the 'system', strongly (at long distances because of asymptotic freedom) interacting with plasma states as the '*environment*'.

2. Entropy and decoherence

(A) We begin by studying the temporal behaviour of the statistical entropy for open systems approaching thermal equilibrium as predicted by the Lindblad equation. General results have been obtained in a recent paper by Olkiewicz [6]:

- If the element Θ_t of the semigroup (2) is contractive in the operator norm, i.e. $\|\Theta_{t_1}\| \leqslant \|\Theta_{t_2}\|$ for $t_2 \leqslant t_1$, then the von Neumann entropy $S = -\mathrm{tr}(\varrho \ln \varrho)$ is non-decreasing, i.e. $S(\Theta_t\varrho) \geqslant S(\varrho)$ for all $\varrho \in \mathfrak{S}(\mathfrak{H})$.[5]
- The semigroup of completely positive maps Θ_t generated by the Lindblad equation is contractive iff

$$\sum_J V_J V_J^+ \leqslant \sum_J V_J^+ V_J$$

  is obeyed by the Lindblad operators V_J defined in (3) and (4).
- In the finite-dimensional case a sufficient condition for monotonic behaviour is that all Lindblad operators be normal: $V_J V_J^+ = V_J^+ V_J$ for all J.

It does not seem clear why these conditions should be obeyed in actual physical situations and, hence, why the entropy should not decrease during the approach to equilibrium (see also [5]) for all times and for any experimental set-up: a scenario in which the entropy transfer between the system and the environment is organized as a *do ut des* intercourse does not seem implausible at all.

To illustrate a situation which is sufficiently general to be interesting we take the one operator model introduced in the introduction and stipulate that our system run in an asymptotic Gibbs state. From equations (10), (14) and (15) we conclude that V has then the form

$$V = U \exp\left(\tfrac{1}{2}\left(\beta H + \beta \sum \alpha_i Q_i\right)\right) \tag{19}$$

where we take U as unitary and assume that it is irreducible[6], β and α_i are the inverse temperature and the chemical potentials respectively. We then arrive at an equation of motion which implies that any initial state of an open system asymptotically approaches a Gibbs distribution; given U, the time dependence of the statistical entropy is then unequivocally predicted. This is in view of the above cited theorems the more interesting since V is neither (operator-) bounded nor contractive. In figures 1–3 we show the result of a numerical computation (for N-dimensional Hilbert spaces) of $S = -\mathrm{tr}(\varrho \ln \varrho)$. The purpose of these

[5] $\mathfrak{S}(\mathfrak{H})$ is the set of all positive, normalized trace class operators, i.e. $\mathbb{C}P^N$ for finite $(N+1)$-dimensional systems.
[6] I.e. $Ns = 1$, see below, in particular equation (22).

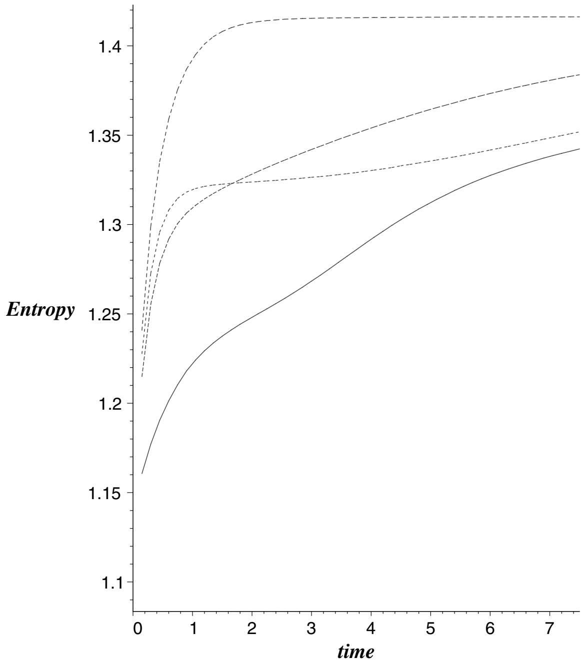

Figure 1. The statistical entropy as a function of time ($N = 4, \beta = 0.1$). The four curves correspond to four different energy spectra and initial conditions chosen at random.

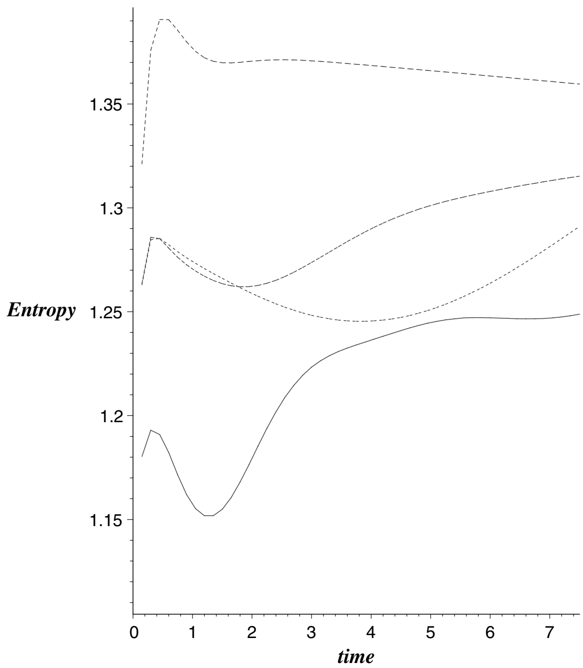

Figure 2. The statistical entropy as a function of time ($N = 4, \beta = 1$). The four curves correspond to four different energy spectra and initial conditions chosen at random.

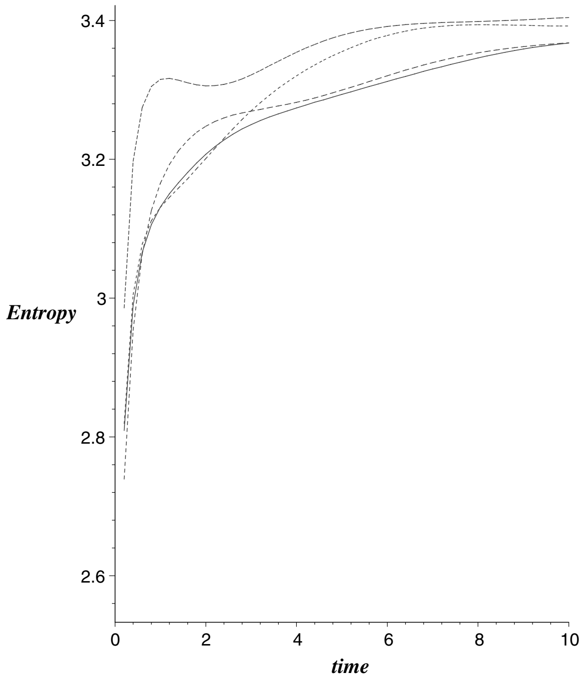

Figure 3. The statistical entropy as a function of time ($N = 10, \beta = 1$). The four curves correspond to four different energy spectra and initial conditions chosen at random.

calculations is to show the occurrence of non-monotonic temporal behaviour of the statistical entropy. In order to avoid specialization to a specific model we propose the following statistical approach:

- We assume a canonical Gibbs distribution with the inverse temperature β.
- The Hamiltonian H, the unitary U in equation (19) and the initial state $\varrho|_{t=0}$ are Hermitian (with eigenvalues in the interval $[1, \ldots, 10]$), unitary and positive, normalized matrices respectively with random entries. The real and imaginary parts of U and $\varrho_{t=0}$ are taken in the interval $[-1, \ldots, 1]$; a time scale is fixed by limiting the eigenvalues of the Hamiltonian to the above interval.

A series of runs (around 50) showed non-monotonic behaviour in all cases; figures 1–3 display some examples.

Of course, all curves approach the asymptotic equilibrium entropy $S = -\varrho_{\text{stationary}} \ln(\varrho_{\text{stationary}})$. Furthermore, it should be observed that the statistical entropy is by no means a monotonically increasing function of time, at least in an early period of the system development.

(B) We now turn to the central problem addressed in this paper, the problem of constructing Lindblad operators V_J such that any initial state of a given system decoheres into a set of independent systems whose composition is fixed to correspond to a specific physical situation. Independence of these asymptotically reached systems is to be understood in the sense of superselection rules, that is to say that pure quantum states of distinct systems do not allow for superposition; a linear superposition formally constructed is not a quantum state of a physically realizable system.

I start by reiterating a result [7] obtained for finite systems: operators V can be constructed such that the equation

$$L^B(B_{\text{stationary}}) = 0$$

separates into a set of independent equations for B. Formally this means that the asymptotic solution appears as a direct sum of operators acting in subspaces $\mathfrak{H}_i$ left invariant by the Lindblad generator: instead of (13) one obtains

$$B \longrightarrow \sum_{\oplus} \frac{\text{trace}_i(\varrho_i B)}{\text{trace}_i(\varrho_i)} \mathbb{I}_i \tag{20}$$

for

$$t \longrightarrow \infty$$

where the index i pertains to these subspaces, the ϱ_i are the projected states and the $\mathbb{I}_i$ are the respective unit operators. The asymptotic state is anticipated as

$$\varrho_{t=0} \longrightarrow \varrho_{\text{stationary}}$$

with

$$\varrho_{\text{stationary}} = \sum_i W_i/\text{trace}_i(W_i) \tag{21}$$

where the W_i are the projections of (10). It should be noted that this relation is a generalization of (14) only as far as the normalizations are concerned: since the equation of motion (4) decays into a set of equations we are free to choose different normalizations in each sector. It is remarkable that the normalization problem is automatically dealt with by equation (4). The trace of the stationary state trace($\varrho_{\text{stationary}}$) $= Ns$ (Ns is the number of invariant subspaces) and not unity as for the initial state trace(ϱ_{initial}) $= 1$ with which the process begins. So $\varrho_{t=0}$ in (21) has to be renormalized

$$\varrho_{t=0} = \varrho_{\text{initial}} + \sum_i (\varrho_{\text{initial},i}/\text{trace}(\varrho_{\text{initial},i}) - \varrho_{\text{initial},i}).$$

Matrix elements of the initial configuration ϱ_{initial} outside the projections $\{i\}$ are transported to zero by the Lindblad motion for $t \to \infty$ and hence are irrelevant for the asymptotic state. These observations could be seen as first indications of a decoherence mechanism. We are now going to prove these assertions in some detail.

We begin by a more precise formulation of the problem. Let

$$\mathfrak{P} = \left\{ P_i \mid P_i P_j = \delta_{ij} P_i,\, P_i = P_i^+,\, \sum_{i=1}^{Ns} P_i = 1,\, i, j = 1 \ldots Ns \text{ (finite)} \right\} \tag{22}$$

be a spectral family (which can be extended to include a (weakly) continuous family of projection operators [8, 9] indexed by measurable sets $\mathbb{D} \subset \mathbb{R}$). Let furthermore

$$\mathfrak{Q} = \{Q_j \mid [Q_i, Q_j] = 0, \text{ for all } i, j; i, j = 1 \ldots\} \tag{23}$$

denote the maximal Abelian subalgebra in $\mathfrak{B}(\mathfrak{H})$ of observables, $H := Q_1$ taken as Hamiltonian. A set of such operators containing H specifies a spectral decomposition

$$H = \sum_i P_i \tilde{H} P_i \tag{24}$$

of the Hamiltonian H; since $\mathfrak{Q}$ is maximal, H can be broken down to a diagonal by simultaneous diagonalization.

The dynamics of the open system is said to induce decoherence into superselection sectors [10], defined by this spectral decomposition if

$$\varrho(t) \longrightarrow \sum_i P_i \varrho_{\text{asymptotic}} P_i \qquad \text{for} \quad t \to \infty \tag{25}$$

or

$$P_i\varrho(t)P_j \longrightarrow 0 \qquad \text{for} \quad t\to\infty \quad \text{and} \quad i\neq j. \tag{26}$$

Again I begin by dealing with the case of only one Lindblad operator which I write as

$$V = \sum V_{i,j}$$

where

$$V_{i,j} = P_i\tilde{V}P_j.$$

The choice

$$V = \sum_i V_{i,\pi[i]} \tag{27}$$

where π is a permutation

$$\pi(1,\ldots,Ns) = (i_1,\ldots,i_{Ns})$$

leads to decoherence (writing in short we identify the result of the operation π with the operation and write $\pi = [i_1,\ldots,i_{Ns}]$ and $\pi_{id} = [1,2,\ldots,Ns]$).

The equation $L^\varrho(\varrho) = 0$ now reads

$$L^\varrho(\varrho) = \sum_i P_i\tilde{V}P_{\pi[i]}\varrho\sum_j P_{\pi[j]}\tilde{V}^+P_j - \frac{1}{2}\left[\sum_i P_{\pi[i]}\tilde{V}^+P_i\tilde{V}P_{\pi[i]},\varrho\right]_+$$

and is seen to couple *mutually and exclusively* the sectors $P_i\varrho P_i$,[7] i.e. the elements of the block-diagonal decomposition of ϱ defined by the given spectral decomposition of H. Inspecting this equation we see from (14) that $\varrho_{\text{stationary}}$ has for *all* choices of π the form

$$\bar{\varrho}_{\text{stationary}} = \sum_i P_iWP_i. \tag{28}$$

Normalizing as indicated above we reproduce (21)

$$\varrho_{\text{stationary}} = \sum_{i=1}^{Ns} P_iWP_i/\text{trace}(P_iWP_i).$$

Equation (20) is now derived in the following manner (see [14]); multiplying (4) from the lhs and the rhs by[8] $((V^+)_{i,i} = (V_{i,i})^+)$

$$\sqrt{W} = \sum_{i=1}^{Ns}\left(V_{i,i}^+V_{i,i}\right)^{-1/2}$$

and taking the trace we see that the rhs of the equation thus obtained vanishes and

$$\text{trace}(BW) = \text{const}$$

follows. Observing that $B \to \text{const}\,\mathbb{I}$ for $t\to\infty$ in each projected sector equation (20) follows by taking this relation at $t = 0$ and $t = \infty$.

An essential element in the formulation of quantum dynamics is the notion of its independence from the choice of basis in the representation space $\mathfrak{H}$. This principle is transferred to the superselection sectors reached via the Lindblad motion (more precisely this transfer turns these sectors into superselection sectors): starting with $U^+\varrho_t|_{t=0}U$ leads us to

[7] That is to say that there are two independent sets of equations separately coupling off-diagonals and diagonals.

[8] $W = \sum_{i=1}^{Ns}(V_{i,i}^+V_{i,i})^{-1}$ and equation (9) reads $[H_i, W_i] = 0$ for all i, where $W_i = P_iWP_i$ and $H_i = P_iHP_i$.

$U^+\varrho_{\text{stationary}}U$ and hence to[9]

$$U \longrightarrow U_{\text{asymptotic}} = \sum_{i=1}^{Ns} P_i U P_i \qquad \text{for} \quad t \to \infty \quad \text{and all unitaries } U \text{ on } \mathfrak{H}. \tag{29}$$

We see therefore that each sector carries its own quantum dynamics.

To see that the decomposition (24) is instrumental for the decoherence mechanism just described we first observe that by construction the second term in the Lindblad generators (6) and (7) is purely absorptive so that only the stationary solution remains at $t = \infty$. Had we introduced an off-diagonal part $H_{l,m} = P_l \tilde{H} P_m$ $(l \neq m)$ in (23) an inhomogeneous term would appear in the equation of motion for this (l, m) sector and hence an asymptotic mixing of the (l, l) and (m, m) sectors would result. A new superselection sector would arise defined on a subspace spanned by linear combinations of vectors in $\mathfrak{H}_l$ and $\mathfrak{H}_m$, the representation spaces of the (l, l) and (m, m) sectors. From our derivation it is clear that the general Lindblad operator

$$L^{\varrho}(\cdot) = \sum_J \left(V_J \cdot V_J^+ - \frac{1}{2}\left[V_J^+ V_J, \cdot\right]_+ \right)$$

has the structure required for decoherence if the V_J all have the same form (26)

$$V_J = \sum_i V_{J(i,\pi[i])} \tag{30}$$

where

$$V_{J(i,j)} = P_i \tilde{V}_J P_j. \tag{31}$$

The equations of motion then have the same structure as in the one-operator case and the same structure of the asymptotic solutions ensues. The choice π is irrelevant for the decoherence pattern as is clear from the above derivation.

The last paragraph of this section is devoted to an explicit construction of the projections employed in our decoherence mechanism. The construction presented pertains to the case of finite (N-)dimensional Hilbert spaces $\mathfrak{H}$ (the infinite case can be treated on similar lines using suitably contrived bijections instead of permutations). For simplicity we start by doing the construction in a basis of $\mathfrak{H}$ in which H and W are simultaneously diagonal (equation (9)). Using the polar decomposition (15) of V we get

$$V = U(W)^{-1/2}$$

where $(W)^{-1/2}$ is a diagonal matrix, U is constructed in the following manner: let e_j, $j = 1, \ldots, N$, $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots), \ldots$ be the usual basis vectors in $\mathfrak{H}$, take a permutation π of $(1, 2, \ldots, N)$ and define the unitary U as

$$U = [e_{\pi[1]}, e_{\pi[2]}, \ldots, e_{\pi[N]}] \tag{32}$$

where the e_i figure as column vectors.

The matrix representations of L^B and L^{ϱ} read

$$M^B = V^+ \otimes V - (V^+V \otimes \mathbb{I} + \mathbb{I} \otimes V^+V)/2$$
$$M^{\varrho} = V \otimes V^+ - (V^+V \otimes \mathbb{I} + \mathbb{I} \otimes V^+V)/2.$$

The role of permutations in determining the structure of stationary configurations has been discussed in [14]; we reiterate and extend these results.

[9] We have $U^+\varrho|_{t=0}U \longrightarrow \sum_i P_i U^+ \varrho_{\text{stationary}} U P_i = \sum_i P_i U^+ P_i W P_i U P_i$.

- The ranks of M^B or M^ϱ (both are of course equal) are determined by the structure of the permutation π. We have

$$\mathrm{Rank}(M^B) = N^2 - n_{\mathrm{int}}$$

where n_{int} is the number of fixpoints in π plus the minimal number of cycles in which cyclic interchanges of two positions in π reduce π to π_{id}. The irreducible case alluded to above is given whenever $n_{\mathrm{int}} = 1$.
- Interchanges are to be performed in terms of cycles. We distinguish 1-, 2-, ..., k-cycles: a 1-cycle is a fixpoint of the permutation, and k denotes the length of the cycle. These cycles determine the structure of the stationary configuration (21) of the state ϱ and the asymptotic observable B (20): the dimensions of the subspaces $\mathfrak{H}_i$ referred to when deriving (20) and (21) are given by the cycle length [14].
- The rank of $M^{\varrho,B}$ is

$$\mathrm{Rank}(M^{\varrho,B}) \leqslant N^2.$$

The minimal rank is obtained in the case of N fixpoints—the identity map, the case of Hermitian V; next to minimal are $N-2$ fixpoints and one interchange. The maximal rank corresponds to no fixpoint and one cyclic interchange. So we have

$$\mathrm{Rank}(M^{\varrho,B})|_{\min} = N^2 - N \qquad \mathrm{Rank}(M^{\varrho,B})|_{\max} = N^2 - 1.$$

- The case of overlapping cycles leads to the following rule: if a cyclic interchange has to be followed by an interchange involving at least one member of the preceding cycle the two cycles are melted into one which then counts for the stationary structure. If the melted cycles are identical with the set π_{id} then

$$\mathrm{Rank}(M_B) = \mathrm{Rank}(M_B)|_{\max} = N^2 - 1.$$

Some examples will illustrate this construction; let $N = 6$ and $\pi_{\mathrm{id}} = [1, 2, 3, 4, 5, 6]$.

- Take $\pi = [2, 3, 4, 5, 6, 1]$ or $[6, 1, 2, 3, 4, 5]$; one interchange in a cycle of length 6 is necessary so that

$$B_{\mathrm{stationary}} = b\mathbb{I}_{\mathfrak{H}}.$$

- Take $\pi = [3, 4, 5, 6, 1, 2]$ (or $[5, 6, 1, 2, 3, 4]$); two cyclic permutations in the two 3-cycles $[1, 3, 5]$ and $[2, 4, 6]$ with the corresponding interchanges $[3, 5, 1] \to [1, 3, 5]$ and $[4, 6, 2] \to [2, 4, 6]$ are required to attain π_{id}. The stationary configuration of B is reducible—not irreducible in the sense explained above—and reads

$$B_{\mathrm{dstat}} = b_1\mathbb{I}_{\mathfrak{H}_{1,3,5}} \oplus b_2\mathbb{I}_{\mathfrak{H}_{2,4,6}}$$

where in obvious notation $\mathfrak{H}_{i-,j-,k}$ denotes the subspace spanned by the i-, j-, k-axes.
- Take $\pi = [4, 5, 6, 1, 2, 3]$, three interchanges in the cycles $[1, 4]$, $[2, 5]$ and $[3, 6]$ are needed in this case and we have

$$B_{\mathrm{stationary}} = b_1\mathbb{I}_{\mathfrak{H}_{1,4}} \oplus b_2\mathbb{I}_{\mathfrak{H}_{2,5}} \oplus b_3\mathbb{I}_{\mathfrak{H}_{3,6}}.$$

- Take $[1, 4, 3, 5, 6, 2]$; two cycles lead to π_{id}: $[2, 6]$ and $[2, 4, 5]$ with the corresponding interchanges $[6, 2] \to [2, 6]$ and $[4, 5, 2] \to [2, 4, 5]$. Together with the two fixed points $[1]$ and $[3]$ the melting of the two cycles implies the stationary structure

$$B_{\mathrm{stationary}} = b_1\mathbb{I}_{\mathfrak{H}_1} \oplus b_2\mathbb{I}_{\mathfrak{H}_3} \oplus b_3\mathbb{I}_{\mathfrak{H}_{2,4,5,6}}.$$

- Take $\pi = [2, 4, 5, 3, 6, 1]$; the cycles involved in the reduction to π_{id} overlap and cover the set π_{id}. In detail we have the interchanges in the cycles $[4, 5, 3] \to [3, 4, 5]$, and $[2, 3, 4, 5, 6, 1] \to \pi_{\mathrm{id}}$ to be performed in this order. We have overlapping cycles and therefore an irreducible M^B and $n_{\mathrm{int}} = 1$.

Summarizing the essence of our findings we have seen that the decomposition into cycles of a given permutation π employed in the construction of U and V provides us with a construction of projectors P_i—$\{\bar{P}_{[1,3,5]}, \bar{P}_{[2,4,6]}\}$, $\{\bar{P}_{[1,4]}, \bar{P}_{[2,5]}, \bar{P}_{[3,6]}\}$, $\{\bar{P}_{[1]}, \bar{P}_{[3]}, \bar{P}_{[2,4,5,6]}\}$, $\{\bar{P}_{[1,2,3,4,5,6]} = \mathbb{I}_{\mathfrak{H}}\}$ in the four examples given above—leading in turn to superselection sectors as found in the formal construction described. The coefficients b_i are the traces given in (20). Leaving the assumption of a basis in $\mathfrak{H}$ such that H and W are simultaneously diagonal we see that the most general case is obtained by transforming this basis with the unitaries $U_{\text{asymptotic}}$ defined in (29).

Of particular interest is the observation that for simultaneously diagonal W, H and $\sqrt{V^+V} = (\delta_{n,m}\sqrt{(n)})$, the Lindblad operator V corresponding to the cyclic permutation $(1, 2, \ldots, N) \to (2, 3, \ldots, N, 1)$ is a finite-dimensional analogue of the customary creation operator a^+ (the first anticyclic permutation leads to the annihilation operator a); higher cycles and anticycles lead to monomials in a^+ and a respectively, a^+a is the *number-operator mod* (N). Thus, for each cycle in the decomposition of π separately, a 'particle' interpretation can be employed in the construction of superselection sectors. Such a choice is however only a special case, applicable only for special physical phenomena—e.g. photon distributions. Thermalization for instance requires a different Lindblad operator (equation (19)).

3. Summary

The input required to make a Lindblad scheme for the dynamics of open systems predictive consists of the Hamiltonian, of course, and of a set of operators, V_J, parametrizing the absorptive part of the generator for the temporal motion of the system. In a model with only one operator V a key for the construction of V is given by the observation that the inverse square of its absolute value $\sqrt{(V^+V)}$ equals the asymptotic state (density operator) reached by the system at $\tau \to \infty$; its positivity and normalization entails a probability interpretation. Hence we know the explicit form of the Lindblad operator up to an isometry U which we take as unitary in our considerations (invertibility of V^+V and VV^+ or finite systems are assumed). Assuming thermal equilibrium for the final state, i.e. a Gibbs distribution, a fairly general class of models remains for which the Hamiltonian H and U parametrizing the absorptive part of the generator determine the motion. I studied the motion of the statistical entropy in toy models acting in finite-dimensional Hilbert spaces with the Hamiltonian H and the unitary matrices U carrying random entries. In particular, some attention to the dependence on the inverse temperature β and on the matrices U has been given although no systematic study of these dependences was attempted. Non-monotonic behaviour in the approach to equilibrium has been observed, a feature which appears to be a characteristic of (one-operator) Lindblad motion. Our numerical results can be interpreted to point to a tendency of this non-monotonic behaviour prevailing for small β. A dependence of this phenomenon on U should also be noted.

Our next project was a study of decoherence in Lindblad dynamics. Decoherence of an open system into superselection sectors is generated by a spectral decomposition of the Hamiltonian and is realized, for Lindblad motion, by the absorptive part of the generator of temporal motion acting as a sort of filter. A general construction expressing the structure of the V_J in terms of the initially chosen spectral decomposition of the Hamiltonian is described. For a finite number of superselection sectors permutations play an essential role and secure a coupling scheme for the equations of motion in which diagonals couple to diagonals: thus the decoherence channels are coupled among themselves. For a infinite or even continuous decoherence pattern permutations should be replaced by a suitable choice of bijections. The superselective nature of the asymptotic states reached by equations of motion constructed in this way should be checked by ensuring independent invariance of expectation values in each sector: we showed that indeed a unitary transformation for the initial state disintegrates by the Lindblad motion into a direct sum of unitaries acting in each superselection sector. Physically speaking this decoherence mechanism opens an interesting route to the question of dynamical symmetry generation. We hope to come back to this problem in a future publication.

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